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CONSERVATION LAWS AND SYMMETRIES FOR HAMILTONIAN SYSTEMS WITH INPUTS

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Abstract

After a brief introduction to Hamiltonian systems with external forces (inputs) we define symmetries and conservations laws for such systems, and prove a generalization of Noether's theorem. Finally we show how this theory can be applied to the solution of optimal control problems.

1. Hamiltonian systems with external forces

First we will briefly review the definition of a Hamiltonian system (with external forces), as pioneered by Brockett [2] (see also Takens [16]) and developed in [7,8,9,10]. Consider a conservative mechanical system with n degrees of freedom, locally represented by n configuration variables q_1, \dots, q_n . Assuming the

dynamical properties of the system to be fully characterized by its Lagrangian function

$L(q_1, \dots, q_n, \dot{q}_1, \dots, \dot{q}_n)$, we obtain the classical Euler-Lagrange equations

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} = F_i, \quad i = 1, \dots, n \quad (1)$$

where $F = (F_1, \dots, F_n)$ is the vector of external forces as measured in the configuration $q = (q_1, \dots, q_n)$. Since the external forces F_i represent the influence of the (unknown) environment on the system they are a priori arbitrary functions of time (if for a certain j , $F_j = -\frac{\partial V}{\partial q_j}(q_j) + u_j$, with u_j arbitrary, then $-\frac{\partial V}{\partial q_j}(q_j)$ is really an *internal* force which should be incorporated into the system by subtracting V from L). Robot manipulators are outstanding examples of the above [11].

In most applications it is not necessary (or not possible) to observe all the configuration variables, but only a *part* of them, say (q_1, \dots, q_m) , $m \leq n$. In this case it is natural to assume that the external forces F_{m+1}, \dots, F_n are all zero. Conversely, as in common practice in mechanical engineering, we may assume that only some of the external forces, say (F_1, \dots, F_m) are nonzero and that we observe the corresponding displacements (q_1, \dots, q_m) . In system theoretical language we then end up with the following set of equations describing a *Lagrangian system*

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} = \begin{cases} u_i & i = 1, \dots, m \\ 0 & i = m+1, \dots, n \end{cases} \quad (2)$$

$$y_j = q_j, \quad j = 1, \dots, m$$

with $u_i = F_i$ the external forces or *inputs*, and $y_i = q_j$ the observations or *outputs*. If we interpret (y_1, \dots, y_m) as coordinates for an m -dimensional output manifold Y , then $(y_1, \dots, y_m, u_1, \dots, u_m)$ can be most naturally interpreted as (local) coordinates for the cotangent bundle T^*Y . This enables us to give a coordinate free definition of the *external power*

$\sum_{j=1}^m u_j(t) \dot{y}_j(t)$. Instead of remaining within the Lagrangian framework we will transfer (2) to its more or less

equivalent (assuming $\frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j}$ to be nonsingular)

Hamiltonian formulation

$$\begin{aligned} \dot{q}_i &= \frac{\partial H}{\partial p_i} & i &= 1, \dots, n \\ \dot{p}_i &= -\frac{\partial H}{\partial q_i} + u_i & i &= 1, \dots, m \\ \dot{p}_i &= -\frac{\partial H}{\partial q_i} & i &= m+1, \dots, n \\ y_j &= q_j & j &= 1, \dots, m \end{aligned} \quad (3)$$

with $p_i = \frac{\partial L}{\partial \dot{q}_i}$ the momenta and $H(q_1, \dots, q_n, p_1, \dots, p_n) = \sum_{i=1}^n \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L(q, \dot{q})$ the internal energy. Notice that the state space M (coordinates (q, p)) as well as the external space T^*Y (coordinates (y, u)) is endowed with a canonically defined Poisson bracket, namely

$$\{F, G\}_M = \sum_{i=1}^n \left(\frac{\partial F}{\partial p_i} \frac{\partial G}{\partial q_i} - \frac{\partial F}{\partial q_i} \frac{\partial G}{\partial p_i} \right), \quad F, G: M \rightarrow \mathbb{R}$$

and

$$\{F^e, G^e\}_{T^*Y} = \sum_{j=1}^m \left(\frac{\partial F^e}{\partial u_j} \frac{\partial G^e}{\partial y_j} - \frac{\partial F^e}{\partial y_j} \frac{\partial G^e}{\partial u_j} \right), \quad F^e, G^e: T^*Y \rightarrow \mathbb{R}$$

If we want to use the full power of Hamiltonian formalism we have to allow for *canonical* coordinate transformations on M ($\phi: M \rightarrow M$ is canonical if $\{F \circ \phi, G \circ \phi\}_M = \{F, G\}_M \circ \phi$, $\forall F, G$). This forces us to generalize (3) to the so-called *affine* Hamiltonian systems [9,10]

$$\begin{pmatrix} \dot{q}_i \\ \dot{p}_i \end{pmatrix} = \begin{pmatrix} \frac{\partial H}{\partial p_i} \\ -\frac{\partial H}{\partial q_i} \end{pmatrix} - \sum_{j=1}^m u_j \begin{pmatrix} \frac{\partial C_j}{\partial p_i} \\ -\frac{\partial C_j}{\partial q_i} \end{pmatrix} \quad i = 1, \dots, n \quad (4)$$

$$y_j = C_j(q, p) \quad j = 1, \dots, m$$

with C_1, \dots, C_m the *output functions* (in (3) we simply had $C_j(q, p) = q_j$). Furthermore a transformation of the outputs (y_1, \dots, y_m) (for instance from Cartesian to angular coordinates) induces a transformation of the external forces (u_1, \dots, u_m) (from translational forces to external torques). This type of canonical transformations on T^*Y which are induced by transformations on Y still leaves the form of equations (4) invariant, but if we allow for general canonical transformations on T^*Y then (4) has to be further generalized to the *general Hamiltonian systems* [8,10]

$$\begin{aligned} \dot{q}_i &= \frac{\partial \tilde{H}}{\partial p_i}(q, p, u) \\ \dot{p}_i &= -\frac{\partial \tilde{H}}{\partial q_i}(q, p, u) \end{aligned} \quad i = 1, \dots, n \quad (5a)$$

$$y_j = -\frac{\partial \tilde{H}}{\partial u_j}(q, p, u) \quad j = 1, \dots, m \quad (5b)$$

To avoid misunderstanding, it is *not* true that (5) can be always reduced to (4) by a canonical transformation on T^*Y , nor can (4) be always reduced to (3) by a canonical transformation on M . Notice that for a given input function $\bar{u}(t) = (\bar{u}_1(t), \dots, \bar{u}_m(t))$ we may think of (5a) as a *time-dependent* Hamiltonian vectorfield [1] and that we can also interpret u in (5a) as a parameter [1]. The set-up sketched above of "dual" variables q en p , y and u , resembles the situation in electrical network theory (currents and voltages, etc.). In fact the *interconnection* of Hamiltonian systems can be defined as in network theory [8,10]. For more motivation for Hamiltonian systems we refer to [2,10].

Now we will proceed to a *coordinate free* definition of an affine or general Hamiltonian system [8,10]. Let the state space M be a symplectic $2n$ -dimensional manifold with symplectic form ω [1]. By Darboux's theorem there exist so-called canonical local coordinates (q, p) such

that $\omega = \sum_{i=1}^n dp_i \wedge dq_i$. The tangent bundle inherits a symplectic form, denoted by $\dot{\omega}$, from ω . Locally, if

$$\omega = \sum_{i=1}^n dp_i \wedge dq_i, \text{ then } \dot{\omega} = \sum_{i=1}^n d\dot{p}_i \wedge dq_i + dp_i \wedge d\dot{q}_i \quad [8,10].$$

Furthermore T^*Y is endowed with its natural symplectic form $\omega^e = \sum_{j=1}^m du_j \wedge dy_j$. Then $\Omega := \pi_1^* \omega - \pi_2^* \omega^e$ is a symplectic form on the product manifold $TM \times T^*Y$ (π_1 and π_2 are the natural projections onto TM , resp. T^*Y). We recall that a submanifold L of a symplectic manifold (N, Ω) is called *Lagrangian* [1] if Ω restricted to L is zero and $\dim L = \frac{1}{2} \dim N$.

Definition 1. A Hamiltonian system $\Sigma(M, T^*Y, L)$, or briefly Σ , is given by a submanifold $L \subset TM \times T^*Y$ such that

- (i) L is a Lagrangian submanifold of $(TM \times T^*Y, \Omega)$
- (ii) L can be parametrized by the coordinates of M and the coordinates of the fibers of T^*Y (the u -coordinates).

Σ is an *affine* Hamiltonian system if moreover

- (iii) the value of the Y -coordinates of a point in L is only a function of the M -coordinates of this point

By condition (ii) the submanifold L describes the set of velocities (\dot{q}, \dot{p}) and outputs y as functions of (q, p, u) , i.e. $\dot{q} = f_q(q, p, u)$, $\dot{p} = f_p(q, p, u)$, $y = h(q, p, u)$.

By (i) L has a *generating function* $\tilde{H}(q, p, u)$ ([1]). This yields equations (5). If (iii) is satisfied the generating function \tilde{H} is affine in u , i.e. of the form

$$H(q, p) - \sum_{j=1}^m u_j C_j(q, p). \text{ This results in (4).}$$

Notice that condition (i) covers two special cases:

1. There are no dynamics, i.e. no state space M . Then L is just a Lagrangian submanifold of T^*Y . This corresponds to a *static* mechanical system.
2. There are no inputs and outputs, i.e. no T^*Y . Then

$L \subset (TM, \dot{\omega})$ corresponds to a (locally) Hamiltonian vectorfield on M .

2. Symmetries and conservation laws.

A symmetry for a Hamiltonian system will consist of *two* parts, a transformation of the state space and a transformation of the external space, which *together* leave the system invariant. For a formal definition we recall the notion of a *prolongation* of a vectorfield. Let S be a vectorfield on M , with integral flow S_t (i.e. $\frac{d}{dt} S_t(x) = S(S_t(x))$). Then $(S_t)_*: TM \rightarrow TM$ is the integral flow of a vectorfield on TM which we denote by \dot{S} .

Definition 2. [7,9,10]. Let $\Sigma(M, T^*Y, L)$ be a Hamiltonian system. A pair (ϕ, ψ) , with $\phi: M \rightarrow M$, $\psi: T^*Y \rightarrow T^*Y$ diffeomorphisms, is called a (Hamiltonian) *symmetry* if

- (i) $\phi^* \omega = \omega$, $\psi^* \omega^e = \omega^e$ (ϕ and ψ are canonical transformations)
- (ii) $(\phi_*, \psi): TM \times T^*Y \rightarrow TM \times T^*Y$ satisfies $(\phi_*, \psi) L = L$.

A pair (S, T) , with S a vectorfield on M , T a vectorfield on T^*Y is called an (infinitesimal Hamiltonian) *symmetry* if

- (i) $L_S \omega = 0$, $L_T \omega^e = 0$ (S and T are (locally) Hamiltonian)
- (ii) The vectorfield (\dot{S}, T) on $TM \times T^*Y$ is tangent to L , i.e. $(\dot{S}, T)(z) \in T_z L$, $\forall z \in L$.

Remark. If we forget about the Hamiltonian structure, i.e. instead of T^*Y we take $Y \times U$, with U an arbitrary input manifold, L is just a submanifold, and conditions (i) are removed, then this is just the definition of an (infinitesimal) symmetry for an arbitrary system with inputs and outputs ([7,8,10]).

It is easy to see that the external part of a symmetry (ψ or the integral flow of T), leaves the external behaviour of the system *invariant*, i.e. if $(y(t), u(t))$, $t \in \mathbb{R}$, is a possible external trajectory

of the system, then so is the time-function

$\psi(y(t), u(t))$, $t \in \mathbb{R}$.

The usual definition of a *conservation law* is that of a function of the state, which remains constant along the trajectories of the system, when no external forces are applied. If external forces are present we modify the definition in the following way. Recall that for

$F: M \rightarrow \mathbb{R}$ its prolongation $\dot{F}: TM \rightarrow \mathbb{R}$ is defined by

$\dot{F}(v) = dF(v)$, $v \in TM$.

Definition 3. [7,9,10]. Let $\Sigma(M, T^*Y, L)$ be a Hamiltonian system. A pair (F, F^e) , with $F: M \rightarrow \mathbb{R}$, $F^e: T^*Y \rightarrow \mathbb{R}$ smooth functions, is a *conservation law* if the function $\dot{F} - F^e: M \times T^*Y \rightarrow \mathbb{R}$ is zero restricted to L .

Remark. In a more suggestive notation this means that $\frac{d}{dt}|_{\text{along } \Sigma} F(q, p) = F^e(y, u)$. Again the above definition can be easily generalized to arbitrary systems.

For Hamiltonian vectorfields it is a basic result that infinitesimal symmetries are in one-to-one correspondence with (local) conservation laws. This is called Noether's theorem (although the original Noether theorem only deals with a particular type of symmetries). In our context we are able to prove a similar correspondence. For a function F on a symplectic manifold we will denote by X_F its corresponding Hamiltonian vectorfield $\Sigma(\frac{\partial F}{\partial p_i}, \frac{\partial}{\partial q_i} - \frac{\partial F}{\partial q_i}, \frac{\partial}{\partial p_i})$

Theorem 4. [7,9,10]. Let $\Sigma(M, T^*Y, L)$ be a Hamiltonian system with an infinitesimal symmetry (S, T) . Then there exists locally a conservation law (F, F^e) . In fact $S = X_F$, $T = X_{F^e}$. Conversely if (F, F^e) is a conservation law for Σ , then (X_F, X_{F^e}) is a symmetry.

Proof. (sketch) (\Rightarrow). Let (S, T) be a symmetry. Then locally there exist, since $L_{S^*}\omega = L_{T^*}\omega^e = 0$, functions F, F^e such that $S = X_F$, $T = X_{F^e}$ [1]. Hence $(\dot{X}_F, \dot{X}_{F^e})$ is tangent to L . Since $\dot{X}_F = X_{\dot{F}}$ (a Hamiltonian vectorfield on $(TM, \dot{\omega})$) and L is Lagrangian it follows that necessarily $\dot{F} - F^e$ is constant on L . By using the freedom in the choice of F^e (determined up to a constant) we obtain that $\dot{F} - F^e$ is zero on L , and hence (F, F^e) is a conservation law. (\Leftarrow) similarly. \square

The above framework is easily extended to groups or algebra's of (infinitesimal) symmetries. The key observation [9,10] is that if (S_i, T_i) , $i = 1, 2$ are symmetries with corresponding conservation laws (F_i, F_i^e) , $i = 1, 2$, then $([S_1, S_2], [T_1, T_2])$ is again a symmetry with the corresponding conservation law $(\{F_1, F_2\}_M, \{F_1^e, F_2^e\}_{T^*Y})$. As noted in [3], symmetry groups can be fruitfully used for the decomposition of a system into smaller subsystems. Especially the existence of an abelian symmetry group for a Hamiltonian system gives rise to an appealing "Hamiltonian decomposition", related to the Jacobi-Liouville theorem ([5], see also [9]).

We notice that (F, F^e) being a conservation law for a Hamiltonian system (5) can also be written as

$$\{\tilde{H}(q, p, u), F(q, p)\} = F^e(y, u) \quad (6)$$

with $y_j = -\frac{\partial \tilde{H}}{\partial u_j}(q, p, u)$. In the case of an affine Hamiltonian system (4) it follows that $F^e(y, u)$ is of the form $\sum_{j=1}^m u_j K_j(y) + V(y)$. Hence we obtain

$$\{H(q, p), F(q, p)\} = V(C_1(q, p), \dots, C_m(q, p)) \quad (7)$$

Under certain regularity conditions it can then be proven [7,10] that there exists (locally) a function $P: Y \rightarrow \mathbb{R}$ such that $\{H + P \circ C, F\} = 0$, and hence F is a first integral for the modified internal energy $H + P \circ C$. This addition of the term $P \circ C$ corresponds to output feedback.

3. Applications to optimal control ([10,12])

Consider an (unrestricted and smooth) Bolza problem of minimizing (w.r.t. $u(\cdot)$) the cost functional

$$J(x_0, u(\cdot)) = K(x(T)) + \int_0^T L(x(t), u(t)) dt \quad (8)$$

under the constraints

$$\dot{x}(t) = f(x(t), u(t)), x(0) = x_0, x \in \mathbb{R}^n, u \in \mathbb{R}^m \quad (9)$$

The Maximum Principle tells us to introduce the Hamiltonian function $H: \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}$ defined as

$$H(x, p, u) := p^T f(x, u) - L(x, u) \quad (10)$$

(with $p \in \mathbb{R}^n$ the *co-states*) and to consider the Hamiltonian system

$$\begin{aligned} \dot{x}_i &= \frac{\partial H}{\partial p_i}(x, p, u) & i &= 1, \dots, n \\ \dot{p}_i &= -\frac{\partial H}{\partial x_i}(x, p, u) \\ y_j &= -\frac{\partial H}{\partial u_j}(x, p, u) & j &= 1, \dots, m \end{aligned} \quad (11)$$

together with boundary conditions $x(0) = x_0$, $p(T) = -\frac{\partial K}{\partial x}(x(T))$. An *optimal* $u^*: [0, T] \rightarrow \mathbb{R}^m$ has to be such that the "outputs" $y_j(t) = -\frac{\partial H}{\partial u_j}(x(t), p(t), u^*(t))$ are all identically zero on $[0, T]$. If we for simplicity assume that $(\frac{\partial^2 H}{\partial u_i \partial u_j})$ is everywhere non-singular then we can (locally) construct the Legendre transform ([10]) of $H(x, p, u)$ w.r.t. u . If this function is denoted by $\bar{H}(x, p, u)$ then the optimal Hamiltonian $H^0(x, p) = H(x, p, u^*(x, p))$ is given by $\bar{H}(x, p, 0)$ ([10]). It is now worthwhile to look for *symmetries* of the Hamiltonian $H^0(x, p)$ (or equivalently the vectorfield X_{H^0}) because of two reasons. In the first place the existence of symmetries makes it easier to solve for the *optimal trajectory*

$$\begin{aligned} \dot{x}^* &= \frac{\partial H^0}{\partial p}(x^*, p^*) \\ \dot{p}^* &= -\frac{\partial H^0}{\partial x}(x^*, p^*) & x^*(0) = x_0, p^*(T) = -\frac{\partial K}{\partial x}(x^*(T)) \end{aligned} \quad (12)$$

since to every symmetry there corresponds a first

integral of the differential equation (12) (Noether). In the second place the computation of the *optimal control* in feedback form $u^*(t) = u^*(x^*(t), t)$ becomes easier. For instance in the linear case ($f(x, u) = Ax + Bu$, $L(x, u) = \frac{1}{2} x^T Q x + \frac{1}{2} u^T R u$) Hamiltonian symmetries for the optimal Hamiltonian vectorfield

$$\begin{pmatrix} \dot{x} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} A & BR^{-1}B^T \\ Q & -A^T \end{pmatrix} \begin{pmatrix} x \\ p \end{pmatrix} \quad (13)$$

are in one-to-one correspondence with symmetries of the associated Riccati-equation

$$\dot{K} = -A^T K - KA - KBR^{-1}B^T K + Q \quad (14)$$

as noted in [15]. This may considerably simplify the solution of (14). For the nonlinear case we refer to [12].

In many cases however it is hard to explicitly construct $H^0(x, p)$. It is then easier to look for symmetries or conservation laws of the *Hamiltonian system* (11). In fact if (F, F^e) is a conservation law for (11), i.e. $\{H(x, p, u), F(x, p)\} = F^e(-\frac{\partial H}{\partial u}, u)$, also satisfying $F^e(y, u) = 0$ for $y = 0$, then it follows that $\{H^0(x, p), F(x, p)\} = 0$ ([10, 12]). Hence we have found a first integral (or symmetry) for the optimal Hamiltonian! A special but important case of this was considered in [4], by looking at conservation laws (F, F^e) for (11) with F^e identically zero and $F(x, p)$ of the form $p^T G(x)$, with $G(x)$ a n -vector.

Remark. The theory of symmetries for Hamiltonian systems can be also applied to filtering problems. For example, symmetries of spectral density matrices (which can be also viewed as Hamiltonian transfer matrices) are studied in [14], while it is not hard to apply the theory to the Hamiltonian system resulting from maximum likelihood estimation in [6].

4. Some Extensions

Up till now we have only dealt with the most common and "down-to-earth" concept of symmetry and conservation law. In (mathematical) physics there has been a tendency to generalize the notion of (especially) symmetry in several directions. Parallel to this development we wish to give some extensions of Definition 2. A minor extension is the following. Instead of looking for pairs of Hamiltonian vectorfields (S, T) such that (\dot{S}, T) is tangent to L we might also look for a general Hamiltonian vectorfield R on (TM, ω) (not necessarily of the form \dot{S}) such that (R, T) is tangent to L . A promising approach seems to be the following. Consider a Hamiltonian system $\Sigma(M, T^*Y, L)$. We can *prolongate* the system as follows. Since L is a Lagrangian submanifold of $(TM \times T^*Y, \dot{\omega} - \omega^e)$, it follows that TL is a Lagrangian submanifold of $(T(TM) \times T(T^*Y), \ddot{\omega} - \dot{\omega}^e)$, where $\ddot{\omega}$ is the symplectic form on $T(TM)$ canonically induced by $\dot{\omega}$ (like $\dot{\omega}$ is induced by ω), and $\dot{\omega}^e$ is induced by ω^e . Since $(T(T^*Y), \dot{\omega}^e)$ is symplectomorphic to $T^*(TY)$ with its natural symplectic form as a cotangent bundle, we have obtained a new Hamiltonian system with state space TM and output manifold T^*Y . Now we can investigate the symmetries (in the sense of Definition 2) for this

prolongated Hamiltonian system. Of course this prolongation procedure can be continued, and we obtain a *hierarchy* of symmetries. These symmetries correspond to what (in the case of differential equations) are usually called *higher-order symmetries*. Needless to say that the above prolongation idea can be also applied to general symmetries for general (not necessarily Hamiltonian) systems.

Another interesting problem is to investigate how the existence of "enough" symmetries for a system is related to the linearizability of the system. We remark that for the Hamiltonian case the conditions for (local) linearizability are derived in [13].

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